

PROPERTIES OF A TWO-SPHERE SINGULARITY

DEBORAH A. KONKOWSKI

*Department of Mathematics, U.S. Naval Academy
Annapolis, Maryland, 21012, USA
E-mail: dak@usna.edu*

THOMAS M. HELLIWELL

*Department of Physics, Harvey Mudd College
Claremont, California, 91711, USA
E-mail: helliwell@HMC.edu*

Recently Böhmer and Lobo have shown that a metric due to Florides can be extended to reveal a classical singularity that has the form of a two-sphere. Here we discuss and expand on the classical singularity properties and then show the classical singularity is not healed by a quantum analysis.

1. Introduction

The question addressed in this review is: What are the properties of the unusual two-sphere singularity discovered by Böhmer and Lobo? The answer is given for quantum as well as classical singularity structure. This conference proceeding is based on articles by Böhmer and Lobo¹ and by the authors.^{2,3}

2. Types of Singularities

2.1. *Classical Singularities*

A classical singularity is indicated by incomplete geodesics or incomplete paths of bounded acceleration^{4,5} in a maximal spacetime. Since, by definition, a spacetime is smooth, all irregular points (singularities) have been excised; a singular point is a boundary point of the spacetime. There are three different types of singularity:⁶ quasi-regular, non-scalar curvature and scalar curvature. Whereas quasi-regular singularities are topological, curvature singularities are indicated by diverging components of the Riemann tensor when it is evaluated in a parallel-propagated orthonormal frame carried along a causal curve ending at the singularity.

2.2. *Quantum Singularities*

A spacetime is QM (quantum-mechanically) nonsingular if the evolution of a test scalar wave packet, representing the quantum particle, is uniquely determined by

the initial wave packet, manifold and metric, without having to put boundary conditions at the singularity.⁷ Technically, a static ST (spacetime) is QM-singular if the spatial portion of the Klein-Gordon operator is not essentially self-adjoint on $C_0^\infty(\Sigma)$ in $L^2(\Sigma)$ where Σ is a spatial slice. This is tested (see, e.g., Konkowski and Helliwell^{2,3,8}) using Weyl's limit point - limit circle criterion^{9,10} that involves looking at an effective potential asymptotically at the location of the singularity. Here a limit-circle potential is quantum mechanically singular, while a limit-point potential is quantum mechanically non-singular.

3. 2-Sphere Singularity – Böhmer-Lobo Space-time

The Böhmer and Lobo metric¹ is

$$ds^2 = -\frac{dt^2}{\cos \alpha} + R^2 d\alpha^2 + R^2 \sin^2 \alpha d\Omega^2. \quad (1)$$

where $R = \sqrt{3/8\pi\rho_0}$ in terms of the constant energy density ρ_0 , and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. The coordinate ranges are $-\infty < t < \infty$, $0 \leq \theta \leq \pi$, and $0 \leq \phi < 2\pi$. The radial coordinate α can either take the values $0 < \alpha \leq \pi/2$ (half a three-sphere) or $-\pi/2 \leq \alpha \leq \pi/2$ (two half three-spheres joined at $\alpha = 0$ with $\alpha = -\pi/2$ identified with $\alpha = +\pi/2$).

The Böhmer-Lobo spacetime is static, spherically symmetric, regular at $\alpha = 0$, and it has vanishing radial stresses.¹ It is also Petrov Type D and Segre Type A1 ([11] 1, 1), and it satisfies the strong energy condition automatically and the dominate energy condition with certain more stringent requirements.¹ Vertical cuts through the three-sphere define latitudinal two-spheres; in particular, the equatorial cut at $\alpha = \pi/2$ is a two-sphere on which scalar polynomial invariants diverge and the tangential pressure diverges as well.

3.1. Classical singularity structure

One can show that the Böhmer-Lobo spacetime is timelike geodesically complete but null geodesically incomplete. The equatorial two-sphere is a weak, timelike, scalar curvature singularity.

3.2. Quantum singularity structure

The Klein-Gordon equation

$$|g|^{-1/2} \left(|g|^{1/2} g^{\mu\nu} \Phi_{,\nu} \right)_{,\mu} = M^2 \Phi \quad (2)$$

for a scalar function Φ has mode solutions of the form

$$\Phi \sim e^{-i\omega t} F(\alpha) Y_{\ell m}(\theta, \phi) \quad (3)$$

for spherically symmetric metrics, where the $Y_{\ell m}$ are spherical harmonics and α is the radial coordinate. The radial function $F(\alpha)$ for the Böhmer-Lobo metric obeys

$$F'' + \left(2 \cot \alpha + \frac{1}{2} \tan \alpha\right) F' + \left[R^2 \omega^2 \cos \alpha - \frac{\ell(\ell+1)}{\sin^2 \alpha} - R^2 M^2\right] F = 0, \quad (4)$$

and square integrability is judged by finiteness of the integral

$$I = \int d\alpha d\theta d\phi \sqrt{\frac{g_3}{g_{00}}} \Phi^* \Phi, \quad (5)$$

where g_3 is the determinant of the spatial metric. A change of coordinates puts the singularity at $x = 0$ and converts the integral and differential equation to the one-dimensional Schrödinger forms $\int dx \psi^* \psi$ and

$$\frac{d^2 \psi}{dx^2} + (E - V) \psi = 0, \quad (6)$$

where $E = R^2 \omega^2$ with a potential that is asymptotically

$$V(x) \sim \frac{R^2 M^2 + \ell(\ell+1)}{x^{2/3}} < \frac{3}{4x^2}. \quad (7)$$

It follows from Theorem X.10 in Reed and Simon⁹ that $V(x)$ is in the limit circle case, so $x = 0$ is a quantum singularity. The Klein-Gordon operator is therefore not essentially self-adjoint. Quantum mechanics fails to heal the singularity.

References

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